E_1-DEGENERATION AND $d'd''$-LEMMA

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Abstract. For a double complex (A, d', d''), we show that if it satisfies the $d'd''$-lemma and the spectral sequence $\{E_r^{p,q}\}$ induced by A does not degenerate at E_0, then it degenerates at E_1. We apply this result to prove the degeneration at E_1 of a Hodge-de Rham spectral sequence on compact bi-generalized Hermitian manifolds that satisfy a version of $d'd''$-lemma.

Keywords: $\partial\overline{\partial}$-lemma, Hodge-de Rham spectral sequence, E_1-degeneration, bi-generalized Hermitian manifold.

1. Introduction

Complex manifolds that satisfy the $\partial\overline{\partial}$-lemma enjoy some nice properties such as they are formal manifolds([DGMS]), their Bott-Chern cohomology, Aeppli cohomology and Dolbeault cohomology are all isomorphic. Compact Kähler manifolds are examples of such manifolds. The Hodge-de Rham spectral sequence $E_*^{*,*}$ of a complex manifold M is built from the double complex $(\Omega^{*,*}(M), \partial, \overline{\partial})$ of complex differential forms which relates the Dolbeault cohomology of M to the de Rham cohomology of M. It is well known that $E_1^{p,q}$ is isomorphic to $H^p(M, \Omega^q)$ and the spectral sequence $E_r^{*,*}$ converges to $H^*(M, \mathbb{C})$. The goal of this paper is to prove an algebraic version of the result that the $\partial\overline{\partial}$-lemma implies the E_1-degeneration of a Hodge-de Rham spectral sequence. The following is our main result.

Theorem 1.1. *If a double complex (A, d', d'') satisfies the $d'd''$-lemma and the spectral sequence $\{E_r^{p,q}\}$ induced by A does not degenerate at E_0, then it degenerates at E_1.*

We define a spectral sequence that is analogous to the Hodge-de Rham spectral sequence of complex manifolds for bi-generalized Hermitian manifolds. Applying result above, we are able to show that for compact bi-generalized Hermitian manifolds that satisfy a version of $\partial\overline{\partial}$-lemma, the sequence degenerates at E_1.

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2. Degeneration of a Hodge-de Rham spectral sequence

Definition 2.1. *A spectral sequence is a sequence of differential bi-graded modules $\{(E_r^{*,*}, d_r)\}$ such that d_r is of degree $(r, 1-r)$ and $E_{r+1}^{p,q}$ is isomorphic to $H^{p,q}(E_r^{*,*}, d_r)$.*

Definition 2.2. *A filtered differential graded module is a $\mathbb{N}$-graded module $A = \oplus_{k=0}^{\infty} A^k$, endowed with a filtration F and a linear map $d : A \to A$ satisfying*

(1) d is of degree 1: $d(A^k) \subset A^{k+1}$;
(2) $d \circ d = 0$;

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(3) *the filtered structure is descending:*

$$A = F^0A \supseteq F^1A \supseteq \cdots \supseteq F^kA \supseteq F^{k+1}A \supseteq \cdots;$$

(4) *the map d preserves the filtered structure: $d(F^kA) \subset F^kA$ for all k.*

For $p, q, r \in \mathbb{Z}$, let

$$\begin{array}{lll} Z_r^{p,q} = \left\{\xi \in F^pA^{p+q} \middle| d\xi \in F^{p+r}A^{p+q+1}\right\}, & Z_\infty^{p,q} = F^pA^{p+q} \cap \ker d \\ B_r^{p,q} = F^pA^{p+q} \cap dF^{p-r}A^{p+q-1}, & B_\infty^{p,q} = F^pA^{p+q} \cap \mathrm{Im}d \\ E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}}, & E_\infty^{p,q} = \frac{F^pA^{p+q}\cap\ker d}{F^{p+1}A^{p+q}\cap\ker d+F^pA^{p+q}\cap\mathrm{Im}d} \end{array}$$

with the convention $F^{-k}A^{p+q} = A^{p+q}$ and $A^{-k} = \{0\}$ for $k \geq 0$. Let $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ be the differential induced by $d : Z_r^{p,q} \to Z_r^{p+r,q-r+1}$.

Throughout this paper, we always assume that $A = \oplus_{p,q\geq 0}A^{p,q}$ is a double complex of vector spaces over some field with two maps $d'_{p,q} : A^{p,q} \to A^{p+1,q}$ and $d''_{p,q} : A^{p,q} \to A^{p,q+1}$ satisfying $d'_{p+1,q}d'_{p,q} = 0, d''_{p,q+1}d''_{p,q} = 0$ and $d'_{p,q+1}d''_{p,q} + d''_{p+1,q}d'_{p,q} = 0$ for all $p, q \geq 0$. To make notation cleaner, we allow p, q to be any integers by defining $A^{p,q} = 0$ for $p < 0$ or $q < 0$.

Let $A^k = \bigoplus_{p+q=k} A^{p,q}$. Define

$$F^pA^k = \bigoplus_{s=p}^{k} A^{s,k-s}$$

For $p > k$, define $F^pA^k = \{0\}$. This gives a descending filtration on A^k.

Let $d = d' + d''$. The double complex (A, d', d'') then defines a filtered differential graded module (A, d, F). Let $\{E_r^{p,q}\}$ be the corresponding spectral sequence. We are interested in the convergence of $E_r^{p,q}$.

Definition 2.3. *Let $\{E_r^{p,q}\}$ be the spectral sequence associated to the double complex (A, d', d''). If $d_s = 0$ for all $s \geq r$, then we say that $\{E_r^{p,q}\}$ or A degenerates at E_r.*

The following simple lemmas will be used frequently.

Lemma 2.4. *If G' is a vector space and $H < G, H < H'$ are subspaces of G', the natural map $\varphi : \dfrac{G}{H} \to \dfrac{G'}{H'}$ is injective if and only if $G \cap H' = H$, and is surjective if and only if $G' = G + H'$.*

Lemma 2.5. *Let $p, q, r \in \mathbb{Z}$. There are inclusions*

$$\cdots \subset B_0^{p,q} \subset B_1^{p,q} \subset \cdots \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset \cdots \subset Z_1^{p,q} \subset Z_0^{p,q} \subset \cdots,$$

$$Z_{r-1}^{p+1,q-1} \subset Z_r^{p,q}\ ,\ \ B_{r+1}^{p+1,q-1} \subset Z_r^{p,q},\ \ d(Z_r^{p-r,q+r-1}) = B_r^{p,q}$$

Definition 2.6. *Let $\alpha_{p,q,r} : E_{r+1}^{p,q} \to \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_r^{p,q}}$ be the map induced by the composition of inclusion and projection, and $\beta_{p,q,r} : E_r^{p,q} \to \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_r^{p,q}}$ be the map induced by the projection.*

Proposition 2.7. *Let $r \in \mathbb{Z}$. Then*

(1) *$d_r = 0$ if and only if $\beta_{p,q,r}$ is an isomorphism for all $p, q \in \mathbb{Z}$.*
(2) *$d_r = 0$ implies that $\alpha_{p,q,r}$ is an isomorphism for all $p, q \in \mathbb{Z}$.*

Proof. (1) We first note that the map $\beta_{p,q,r}$ is always surjective. By Lemma 2.4, $\beta_{p,q,r}$ is an isomorphism if and only if $Z_r^{p,q} \cap (Z_{r-1}^{p+1,q-1} + B_r^{p,q}) = Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$, or equivalently, $B_r^{p,q} \subseteq Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$. The map $d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}$ is the zero map if and only if $\mathrm{Im} d_r^{p-r,q+r-1} = \{0\}$. This is equivalent to $d(Z_r^{p-r,q+r-1}) = B_r^{p,q} \subseteq Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$, which is equivalent to $\beta_{p,q,r}$ being an isomorphism.

(2) We recall that the isomorphism $E_{r+1}^{p,q} \xrightarrow{\cong} H^{p,q}(E_r^{*,*}, d_r)$ (see [M, Proof of Theorem 2.6]) is induced from some canonical projections and inclusions. If $d_r = 0$, $H^{p,q}(E_r^{*,*}, d_r) \cong E_r^{p,q}$ and we have a commutative diagram

$$\begin{tikzcd}
E_{r+1}^{p,q} \arrow[rr, "\cong"] \arrow[dr, "\alpha_{p,q,r}"'] & & E_r^{p,q} \arrow[dl, "\beta_{p,q,r}"] \\
& \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_r^{p,q}} &
\end{tikzcd}$$

By (1), $\beta_{p,q,r}$ is an isomorphism and hence $\alpha_{p,q,r}$ is an isomorphism. □

Definition 2.8. *Fix a pair of integers (p,q). For nonzero $\xi = \sum_i \xi_i \in \bigoplus_{i\geq 0} A^{p+i,q-i}$ where $\xi_i \in A^{p+i,q-i}$, let $i_0 = \min_i\{\xi_i \neq 0\}$. We call ξ_{i_0} the leading term of ξ and denote it as $\ell^{p,q}(\xi)$. We define $\ell^{p,q}(0) = 0$. For $r \geq 1$, $p, q \in \mathbb{Z}$, let*

$$\mathcal{E}_r^{p,q} := \left\{ \xi = \xi_0 + \xi_1 + \cdots + \xi_{r-1} \middle| \xi_i \in A^{p+i,q-i}, d\xi = d'\xi_{r-1} \notin \mathrm{Im} d'', \ell^{p,q}(\eta) \neq \xi_0 \text{ for all } d\text{-closed } \eta \right\}$$

and

$$\mathcal{E}_0^{p,q-1} := B_0^{p,q} - (Z_{-1}^{p+1,q-1} + B_{-1}^{p,q})$$

Lemma 2.9. *Fix $r_0 \geq 1$.*

(1) If the map $\alpha_{p,q,r}$ is an isomorphism for all $p, q \in \mathbb{Z}, r \geq r_0$, then $\mathcal{E}_r^{p,q} = \emptyset$ for all $p, q \in \mathbb{Z}$, $r \geq r_0$.

(2) If the map α_{p,q,r_0} is not an isomorphism, then $\mathcal{E}_{r_0}^{p,q} \neq \emptyset$.

Proof. Note that by Lemma 2.4, the surjectivity of $\alpha_{p,q,r}$ is equivalent to the condition

$$Z_r^{p,q} = Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} + B_r^{p,q} = Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}.$$

(1) Suppose that $\alpha_{p,q,r}$ is an isomorphism for all $r \geq r_0$. Then $Z_i^{p,q} = Z_{i+1}^{p,q} + Z_{i-1}^{p+1,q-1}$ for all $i \geq r_0$. Assume that $\mathcal{E}_r^{p,q} \neq \emptyset$ for some $r \geq r_0$, $p, q \in \mathbb{Z}$. Let $\xi \in \mathcal{E}_r^{p,q}$. By definition, $Z_{q+2}^{p,q} = Z_{q+3}^{p,q} = \cdots = Z_\infty^{p,q}$. So we may take $j > r$ such that $Z_j^{p,q} = Z_\infty^{p,q}$. Note that $\xi \in Z_r^{p,q}$. Using the relation above, we may write $\xi = \eta_1 + \eta_2$ where $\eta_1 \in Z_j^{p,q}$, $\eta_2 \in Z_{j-2}^{p+1,q-1} + \cdots + Z_{r-1}^{p+1,q-1}$. Since $\ell^{p,q}(\xi) \neq 0$, by comparing the degrees of both sides of $\xi = \eta_1 + \eta_2$, we have $\ell^{p,q}(\xi) = \ell^{p,q}(\eta_1)$. But $d\eta_1 = 0$ which contradicts to the fact that $\ell^{p,q}(\xi)$ is not the leading term of any d-closed element.

(2) Fix $r \geq 1$. Suppose that $\alpha_{p,q,r}$ is not an isomorphism, then $Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} \subsetneq Z_r^{p,q}$.
Let

$$\xi = \xi_0 + \xi_1 + \cdots + \xi_k \in Z_r^{p,q} - (Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}) \text{ where } \xi_i \in A^{p+i,q-i}.$$

If $k > r-1$, let $\xi' = \xi_r + \xi_{r+1} + \cdots + \xi_k \in F^{p+r}A^{p+q} \subset F^{p+1}A^{p+q}$. We have

$$d\xi' = d\xi_r + \cdots + d\xi_k \in F^{p+r}A^{p+q+1} = F^{(p+1)+(r-1)}A^{(p+1)+(q-1)+1}$$

which means that $\xi' \in Z_{r-1}^{p+1,q-1}$. Let $\xi'' = \xi - \xi'$. If $\xi'' \in Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$, then $\xi = \xi' + \xi'' \in Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$ which contradicts to our assumption. Therefore $\xi'' = \xi_0 + \cdots + \xi_{r-1} \in Z_r^{p,q} - (Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1})$. Hence we may assume $\xi = \xi_0 + \cdots + \xi_{r-1}$.

(i) Since $\xi \in Z_r^{p,q}$, by definition, $d\xi \in F^{p+r}A^{p+q+1}$. But $d(\xi_0 + \cdots + \xi_{r-2}) + d''\xi_{r-1} \in A^{p,q+1} \oplus A^{p+1,q} \oplus \cdots \oplus A^{p+r-1,q-r+2}$. This forces $d(\xi_0 + \cdots + \xi_{r-2}) + d''\xi_{r-1} = 0$ and hence $d\xi = d'\xi_{r-1}$.

(ii) If $d'\xi_{r-1} = d''\eta_r$ for some $\eta_r \in A^{p+r,q-r}$, then $d(\xi - \eta_r) = d'\xi_{r-1} - d'\eta_r - d''\eta_r = -d'\eta_r \in A^{p+r+1,q-r} \subset F^{p+(r+1)}A^{p+q+1}$. Hence $\xi - \eta_r \in Z_{r+1}^{p,q}$. Since $\eta_r \in F^pA^{p+q}$ and $d\eta_r \in A^{p+r,q-r+1} \oplus A^{p+r+1,q-r} \subset F^{(p+1)+(r-1)}A^{p+q+1}$, we have $\eta_r \in Z_{r-1}^{p+1,q-1}$. Therefore $\xi = (\xi - \eta_r) + \eta_r \in Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$ which is a contradiction. Hence $d'\xi_{r-1} \notin \mathrm{Im} d''$.

(iii) If ξ_0 is the leading term of a d-closed form $\tau \in F^pA^{p+q}$, then $\xi - \tau \in F^{p+1}A^{p+q}$ and $d(\xi - \tau) = d\xi \in F^{p+r}A^{p+q+1} = F^{(p+1)+(r-1)}A^{p+q+1}$. Hence $\xi - \tau \in Z_{r-1}^{p+1,q-1}$. Then $\xi = \tau + (\xi - \tau) \in Z_\infty^{p,q} + Z_{r-1}^{p+1,q-1} \subset Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$ which is a contradiction.

Hence $\xi \in \mathcal{E}_r^{p,q}$.

$\square$

Lemma 2.10. *(1) $\mathcal{E}_0^{p,q-1} = \emptyset$ if and only if $\beta_{p,q,0}$ is an isomorphism.*

(2) For $r \geq 1$, if $\mathcal{E}_r^{p-r,q+r-1} = \emptyset$, then $\beta_{p,q,r}$ is an isomorphism.

(3) For $r \geq 1$, if $\mathcal{E}_r^{p-r,q+r-1} \neq \emptyset$, then $\beta_{p,q,j}$ is not an isomorphism for $j = 1$ or r.

Proof. We note that $\beta_{p,q,r}$ is an isomorphism if and only if $B_r^{p,q} \subset Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$.

(1) This follows from the definition.

(2) Assume that $\beta_{p,q,r}$ is not an isomorphism. Then there exists $\xi \in B_r^{p,q} - (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q})$. So $\xi = d\eta$ for some $\eta \in F^{p-r}A^{p+q-1}$. Let

$$\eta = \eta_0 + \eta_1 + \cdots + \eta_k \text{ where } \eta_i \in A^{p-r+i,q+r-i-1}.$$

If $k \geq r$, let $\eta' = \eta_r + \cdots + \eta_k \in F^pA^{p+q-1} \subset F^{p-(r-1)}A^{p+q-1}$. Then $d\eta' \in F^pA^{p+q} \cap d(F^{p-(r-1)}A^{p+q-1}) = B_{r-1}^{p,q}$. If $d(\eta - \eta') \in Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$, then $\xi = d(\eta - \eta') + d\eta' \in Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$ which is a contradiction. So $d(\eta - \eta') \in B_r^{p,q} - (Z_{r-1}^{p+1,q} + B_{r-1}^{p,q})$. Hence we may assume $\xi = d\eta$ where $\eta = \eta_0 + \cdots + \eta_{r-1}$.

(i) Comparing the degrees of ξ and $d\eta$, we see that $d\eta = d'\eta_{r-1}$.

(ii) If $\eta_0 = 0$, then $\xi = d(\eta_1 + \cdots + \eta_{r-1}) \in F^pA^{p+q} \cap d(F^{p-(r-1)}A^{p+q-1}) = B_{r-1}^{p,q}$ which is a contradiction. So $\eta_0 \neq 0$.

(iii) If η_0 is the leading term of a d-closed form η'', $\eta - \eta'' \in F^{p-r+1}A^{p+q-1}$ and $\xi = d\eta = d(\eta - \eta'') \in d(F^{p-(r-1)}A^{p+q-1}) \cap F^pA^{p+q} = B_{r-1}^{p,q}$ which is a contradiction. Hence η_0 is not the leading term of any d-closed form.

(iv) If $d'\eta_{r-1} \in \mathrm{Im} d''$, $\xi = d\eta = d'\eta_{r-1} = -d''\eta_r$ for some $\eta_r \in A^{p,q-1}$, then $\xi = d'\eta_r - d\eta_r \in Z_\infty^{p+1,q-1} + B_0^{p,q} \subset Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$ which is a contradiction. Hence $d'\eta_{r-1} \notin \mathrm{Im} d''$.

Therefore, $\eta \in \mathcal{E}_r^{p-r,q+r-1}$.

(3) Assume that $\mathcal{E}_r^{p-r,q+r-1} \neq \emptyset$. Let $\eta = \eta_0 + \cdots + \eta_{r-1} \in \mathcal{E}_r^{p-r,q+r-1}$ where $\eta_i \in A^{p-r+i,q+r-i-1}$. Since $d\eta \in B_r^{p,q}$, if $d\eta \notin Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$, $\beta_{p,q,r}$ is not an isomorphism. So we may assume $d\eta = d'\eta_{r-1} = \xi' + d\eta'$ where $\xi' \in Z_{r-1}^{p+1,q-1}$ and $d\eta' \in B_{r-1}^{p,q}$. Let $\eta' = \eta_1' + \eta_2' + \cdots + \eta_l'$, where

$\eta'_i \in A^{p-r+i,q+r-1-i}$. The degree of $d'\eta_{r-1}$ is (p,q), so by comparing degrees of both sides of $d'\eta_{r-1} = \xi' + d\eta'$, we get

$$d'\eta_{r-1} = d'\eta'_{r-1} + d''\eta'_r \text{ and } d''\eta'_{r-1} = 0.$$

If $d'\eta'_{r-1} \in \mathrm{Im} d''$, then $d'\eta_{r-1} \in \mathrm{Im} d''$ which contradicts to the fact that $\eta \in \mathcal{E}_r^{p-r,q+r-1}$. So $d'\eta'_{r-1} \notin \mathrm{Im} d''$. Note that if η'_{r-1} is the leading term of a d-closed element τ, we may write $\tau = \eta'_{r-1} + \tau_r + \cdots + \tau_k$ for some $k > r-1$ and each $\tau_i \in A^{p-r+i,q+r-1-i}$. Then comparing the degrees of $d'\tau = -d''\tau$, we get $d'\eta_{r-1} = -d''\tau_r$ which contradicts to the fact that $d'\eta_{r-1} \notin \mathrm{Im} d''$.

From the above verification, we see that $\eta'_{r-1} \in \mathcal{E}_1^{p-1,q}$. Assume that $d\eta'_{r-1} \in Z_0^{p+1,q-1} + B_0^{p,q}$. Write $d\eta'_{r-1} = \gamma + d\sigma$ where $\gamma = \gamma_1 + \gamma_2 + \cdots \in Z_0^{p+1,q-1}$, $\gamma_i \in A^{p+i,q-i}$, $\sigma = \sigma_0 + \sigma_1 + \cdots \in B_0^{p,q}$ and $\sigma_i \in A^{p+i,q-1-i}$. Since the degree of $d\eta'_{r-1}$ is (p,q), comparing the degrees of both sides of $d\eta'_{r-1} = \gamma + d\sigma$, we get $d\eta'_{r-1} = d''\sigma_0$ which contradicts to the fact that $\eta'_{r-1} \in \mathcal{E}_1^{p-1,q}$. Therefore $d\eta'_{r-1} \notin Z_0^{p+1,q-1} + B_0^{p,q}$ and hence $\beta_{p,q,1}$ is not an isomorphism.

□

Theorem 2.11. *Suppose that* $(A = \oplus_{p,q\geq 0} A^{p,q}, d', d'')$ *is a double complex and* $r \geq 1$. *The spectral sequence* $\{E_r^{p,q}\}$ *induced by A degenerates at* E_r *but not at* E_{r-1} *if and only if the following conditions hold:*

(1) $\mathcal{E}_k^{p,q} = \emptyset$ *for all* $p,q \in \mathbb{Z}, k \geq r$ *and*
(2) $\mathcal{E}_{r-1}^{p,q} \neq \emptyset$ *for some* p,q.

Proof. Suppose that $\{E_r^{p,q}\}$ degenerates at E_r but not at E_{r-1} for some $r \geq 1$. By Proposition 2.7(2), $\alpha_{p,q,i}$ is an isomorphism for all $p,q \in \mathbb{Z}, i \geq r$. Then by Lemma 2.9, $\mathcal{E}_i^{p,q} = \emptyset$ for all $p,q \in \mathbb{Z}, i \geq r$. Since $d_{r-1} \neq 0$, by Proposition 2.7(1), there are some $p,q \in \mathbb{Z}$ such that $\beta_{p,q,r-1}$ is not an isomorphism. Then by Lemma 2.10, $\mathcal{E}_{r-1}^{p-r+1,q+r-2} \neq \emptyset$.

Conversely, suppose that (1) and (2) hold. By Lemma 2.10, $\beta_{p,q,k}$ is an isomorphism for all $p,q \in \mathbb{Z}, k \geq r$. Then by Proposition 2.7, $d_k = 0$ for $k \geq r$. For the case $r = 1$, by definition, $\mathcal{E}_0^{p,q} \neq \emptyset$ implies that $\beta_{p,q+1,0}$ is not an isomorphism. And hence by Proposition 2.7, $d_0 \neq 0$. For the case $r \geq 2$, if $\beta_{p,q,r-1}$ is an isomorphism for all $p,q \in \mathbb{Z}$, by Proposition 2.7, $d_{r-1} = 0$. Then we have $d_k = 0$ for $k \geq r-1$. By the proof above, $\mathcal{E}_k^{p,q} = \emptyset$ for $k \geq r-1$. In particular, $\mathcal{E}_{r-1}^{p,q} = \emptyset$ for all $p,q \in \mathbb{Z}$ which contradicts to our assumption (2). Therefore there exist some p_0, q_0 such that $\beta_{p_0,q_0,r-1}$ is not an isomorphism. By Proposition 2.7, $d_{r-1} \neq 0$. □

Definition 2.12. *We say that a double complex* (A, d', d'') *satisfies the* $d'd''$*-lemma at* (p,q) *if*

$$\mathrm{Im} d' \cap \ker d'' \cap A^{p,q} = \ker d' \cap \mathrm{Im} d'' \cap A^{p,q} = \mathrm{Im} d'd'' \cap A^{p,q}$$

and A satisfies the $d'd''$*-lemma if A satisfies the* $d'd''$*-lemma at* (p,q) *for all* (p,q).

Now we can give a proof of the main result Theorem 1.1.

Proof. Note that by definition, $d'd''$-lemma implies that $\mathrm{Im} d' \cap \ker d'' \cap A^{p,q} = \mathrm{Im} d' \cap \mathrm{Im} d'' \cap A^{p,q}$ for all p,q. Since $\{E_r^{p,q}\}$ does not degenerate at E_0, $\beta_{p,q,0}$ is not an isomorphism for some p,q, hence by Lemma 2.10, $\mathcal{E}_0^{p,q-1} \neq \emptyset$. Assume that $\mathcal{E}_r^{p,q} \neq \emptyset$ for some $p,q \in \mathbb{Z}$, $r \geq 1$. Then there is $\alpha = \sum_{i=0}^{r-1} \alpha_i \in \mathcal{E}_r^{p,q}$ where $\alpha_i \in A^{p+i,q-i}$. From the condition $d\alpha = d'\alpha_{r-1}$, we have $d''\alpha_{r-1} = -d'\alpha_{r-2}$ and hence $d''d\alpha = -d'd''\alpha_{r-1} = 0$. So $d\alpha = d'\alpha_{r-1} \in (\mathrm{Im} d' \cap \ker d'') \cap A^{p,q} = (\mathrm{Im} d' \cap \mathrm{Im} d'') \cap A^{p,q}$. But by the definition of $\mathcal{E}_r^{p,q}$, $d'\alpha_{r-1} \notin \mathrm{Im} d''$ which leads to a contradiction. Therefore by Theorem 2.11, $\{E_r^{p,q}\}$ degenerates at E_1. □

In the following, we apply the main result to prove the E_1-degeneration of a spectral sequence of bi-generalized Hermitian manifolds. We refer the reader to [G1, C] for generalized complex geometry, and to [CHT] for bi-generalized complex manifolds. We give a brief recall here. A bi-generalized complex structure on a smooth manifold M is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ where $\mathcal{J}_1, \mathcal{J}_2$ are commuting generalized complex structures on M. A bi-generalized complex manifold is a smooth manifold M with a bi-generalized complex structure. A bi-generalized Hermitian manifold $(M, \mathcal{J}_1, \mathcal{J}_2, \mathbb{G})$ is an oriented bi-generalized complex manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ with a generalized metric $\mathbb{G}$ which commutes with $\mathcal{J}_1$ and $\mathcal{J}_2$. We define

$$U^{p,q} := U_1^p \cap U_2^q$$

where $U_1^p, U_2^q \subset \Gamma(\Lambda^* \mathbb{T}M \otimes \mathbb{C})$ are eigenspaces of $\mathcal{J}_1, \mathcal{J}_2$ associated to the eigenvalues ip and iq respectively and $\mathbb{T}M = TM \oplus T^*M$ is the generalized tangent space. It can be shown that the exterior derivative d is an operator from $U^{p,q}$ to $U^{p+1,q+1} \oplus U^{p+1,q-1} \oplus U^{p-1,q+1} \oplus U^{p-1,q-1}$ and we write

$$\delta_+ : U^{p,q} \to U^{p+1,q+1}, \delta_- : U^{p,q} \to U^{p+1,q-1}$$

for the projection of d into corresponding spaces.

Definition 2.13. *On a bi-generalized Hermitian manifold M, there is a double complex $\{(A, d', d'')\}$ given by*

$$A^{p,q} := U^{p+q,p-q}, d' = \delta_+, d'' = \delta_-$$

We call the spectral sequence $\{E_*^{*,*}\}$ associated to this double complex the ∂_1-Hodge-de Rham spectral sequence.

By Theorem 1.1, we have the following result.

Theorem 2.14. *Suppose that M is a compact bi-generalized Hermitian manifold which satisfies the $\delta_+\delta_-$-lemma and has positive dimension. Then the ∂_1-Hodge-de Rham spectral sequence degenerates at E_1.*

Now we give a proof of the E_1-degeneration of the ∂_1-Hodge-de Rham spectral sequence.

Proof. Since $\bigoplus_{p,q} U^{p,q} = \Omega^\bullet(M) \otimes \mathbb{C}$ (see [Ca07], pg 36) where $\Omega^\bullet(M)$ is the collection of smooth forms on M, some $U^{p,q}$ is not empty. The space $U^{p,q}$ is a $C^\infty(M, \mathbb{C})$-module where $C^\infty(M, \mathbb{C})$ is the ring of complex-valued smooth functions on M, and M has positive dimension, therefore $U^{p,q}$ is an infinite dimensional complex vector space . If δ_- is a zero map, we have $H^{p,q}_{\delta_-}(M) = U^{p,q}$ for all p, q. But M is compact, this contradicts to the fact that $H^{p,q}_{\delta_-}(M)$ is finite dimensional([CHT, Theorem 2.14, Corollary 3.11]). Hence δ_- is not the zero map. and the spectral sequence does not degenerate at E_0. Since we assume that M satisfies the $\delta_+\delta_-$-lemma, by Theorem 1.1, the spectral sequence degenerates at E_1. ◻

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